Lecture 1. Linear systems

Def A <u>linear equation</u> in variables $x_1, x_2, \cdots, x_n$ is an equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $\underbrace{a_1, a_2, \cdots, a_n}_{\text{"coefficients"}}$ and b are constants.

e.g. $2x - 3y = 4$, $7x_1 - \frac{3}{2}x_3 = \sqrt{2}$, $x_1 = 3x_2 - 5x_3 + 1$

$\rightsquigarrow x_1 - 3x_2 + 5x_3 = 1$

Note Most equations are not linear.

e.g. $xy + z = 3$, $x_1^2 + 3x_2 - 2x_3 = 4$, $x - 3e^y = 1$

Def A <u>linear system</u> is a collection of linear equations

e.g. $$\begin{cases} 3x_1 - 2x_2 + x_3 = 3 \\ 4x_1 + x_2 - 3x_3 = 0 \\ x_2 + 2x_3 = 1 \end{cases}$$

Note Given a linear system, we aim to answer the following questions:

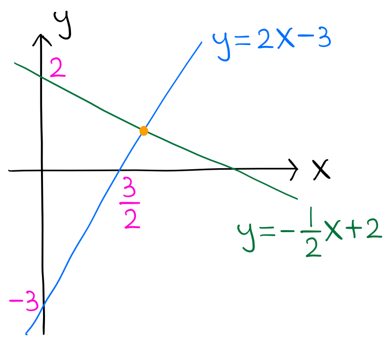
- Does it have a solution?
- If it has solutions, can we find them all?

Ex Determine whether each linear system has a solution.

(1) $\begin{cases} x+2y=4 \\ 2x-\ \ y=3 \end{cases}$

Sol The equation $y=mx+b$ represents the line on the xy-plane with slope m and y-intercept b.

$$\begin{cases} x+2y=4 \\ 2x-\ \ y=3 \end{cases} \Rightarrow \begin{cases} 2y=\quad x+4 \\ -y=-2x+3 \end{cases} \Rightarrow \begin{cases} y=\frac{1}{2}x+2 \\ y=\ 2x-3 \end{cases}$$



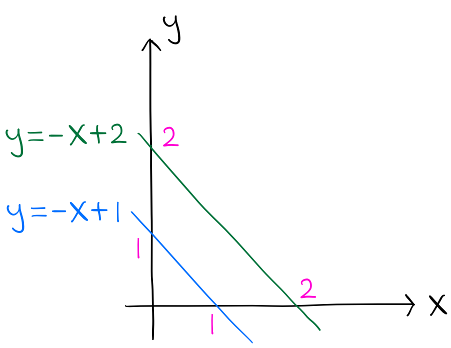
The lines are not parallel (different slopes)

$\Rightarrow$ They intersect at a point

$\Rightarrow$ The system has $\boxed{\text{a unique solution}}$

(2) $\begin{cases} 2x+2y=2 \\ 3x+3y=6 \end{cases}$

Sol $\begin{cases} 2x+2y=2 \\ 3x+3y=6 \end{cases} \Rightarrow \begin{cases} 2y=-2x+2 \\ 3y=-3x+6 \end{cases} \Rightarrow \begin{cases} y=-x+1 \\ y=-x+2 \end{cases}$



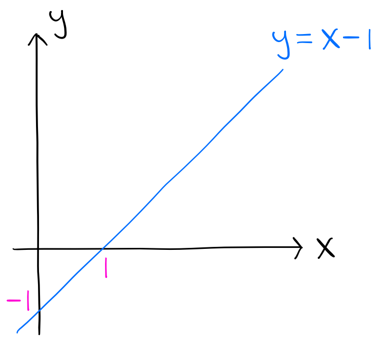
The lines are parallel (same slopes)

$\Rightarrow$ They do not intersect

$\Rightarrow$ The system has $\boxed{\text{no solutions}}$

(3) $\begin{cases} x - y = 1 \\ 2x - 2y = 2 \end{cases}$

$\underline{\text{Sol}}$ $\begin{cases} x - y = 1 \\ 2x - 2y = 2 \end{cases} \Rightarrow \begin{cases} -y = -x + 1 \\ -2y = -2x + 2 \end{cases} \Rightarrow \begin{cases} y = x - 1 \\ y = x - 1 \end{cases}$



The lines coincide

$\Rightarrow$ Every point on the line yields a solution

$\Rightarrow$ The system has $\boxed{\text{infinitely many solutions}}$

<u>Note</u> For a linear system of 2 equations in 2 variables, we have 3 possibilities for the number of its solutions.

- a unique solution (nonparallel lines)
- no solutions (parallel lines)
- infinitely many solutions (same lines)

We will learn in Lecture 3 that the same possibilities apply for all linear systems.

Ex Find the solution of each linear system.

(1) $\begin{cases} 3x_1 + 2x_2 = 7 & (\text{Eq. } 1) \\ \quad\quad\; 3x_2 = 6 & (\text{Eq. } 2) \end{cases}$

Sol (Eq. 2): $3x_2 = 6 \Rightarrow \underline{x_2 = 2}$

(Eq. 1): $3x_1 + 2x_2 = 7 \Rightarrow 3x_1 + 2\cdot 2 = 7 \Rightarrow 3x_1 = 3 \Rightarrow x_1 = 1$

Hence the solution is given by $\boxed{x_1 = 1,\ x_2 = 2}$

(2) $\begin{cases} x_1 - 2x_2 = 1 & (\text{Eq. } 1) \\ 2x_1 + x_2 = 7 & (\text{Eq. } 2) \end{cases}$

Sol We replace (Eq. 2) with a new equation which does not involve x_1.

(Eq. 2) − 2(Eq. 1): $(2x_1 + x_2) - 2(x_1 - 2x_2) = 7 - 2\cdot 1$

$\Rightarrow \cancel{2x_1} + x_2 - \cancel{2x_1} - 4x_2 = 5$ (x_1 eliminated)

$\Rightarrow 5x_2 = 5 \Rightarrow \underline{x_2 = 1}$

(Eq. 1): $x_1 - 2x_2 = 1 \Rightarrow x_1 - 2\cdot 1 = 1 \Rightarrow x_1 = 3$

Hence the solution is given by $\boxed{x_1 = 3,\ x_2 = 1}$

(3) $\begin{cases} 2x + 4y = 10 & (\text{Eq. } 1) \\ 3x - 5y = 4 & (\text{Eq. } 2) \end{cases}$

Sol $\frac{1}{2}$(Eq. 1): $\frac{1}{2}(2x + 4y) = \frac{1}{2}\cdot 10 \Rightarrow x + 2y = 5$ (Eq. 1R)

coefficients 1

(Eq. 2) − 3(Eq. 1R): $(3x - 5y) - 3(x + 2y) = 4 - 3\cdot 5$

$\Rightarrow \cancel{3x} - 5y - \cancel{3x} - 6y = -11$ (x eliminated)

$\Rightarrow -11y = -11 \Rightarrow \underline{y = 1}$

(Eq. 1R): $x + 2y = 5 \Rightarrow x + 2\cdot 1 = 5 \Rightarrow x = 3$

Hence the solution is given by $\boxed{x = 3,\ y = 1}$